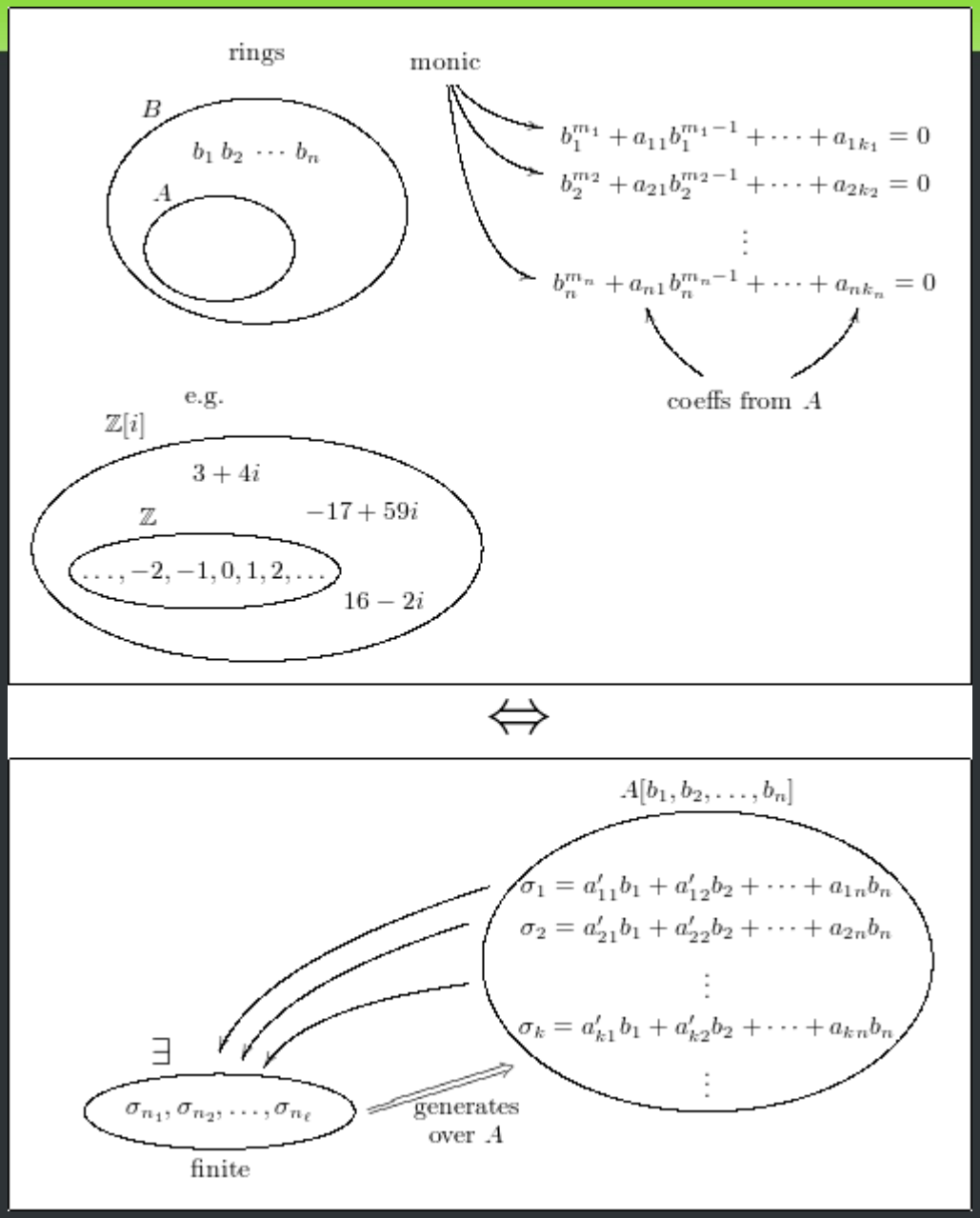
Illustrating Mathematics



- Steve Kieffer
- Simon Fraser University

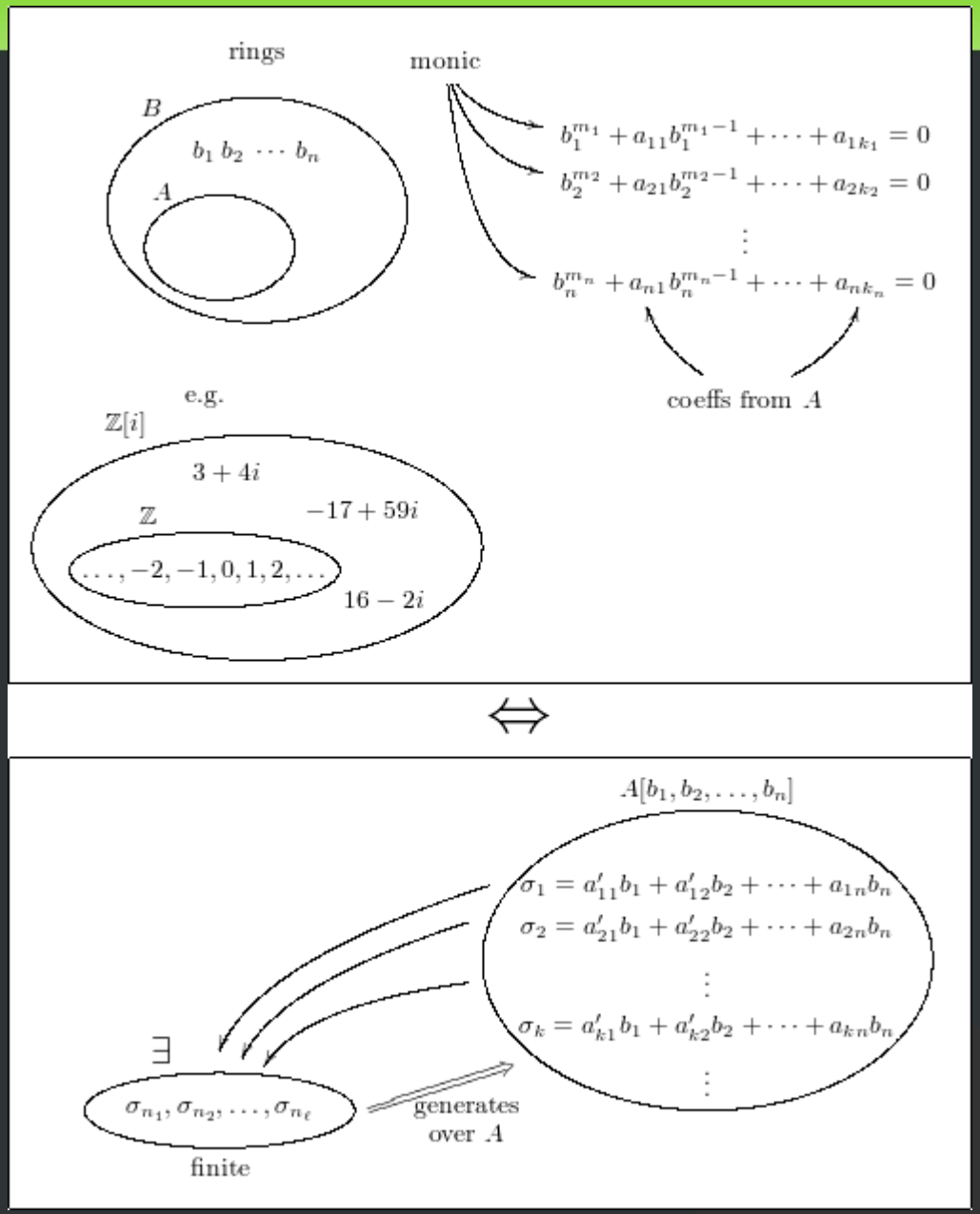
The problem:

Transform this

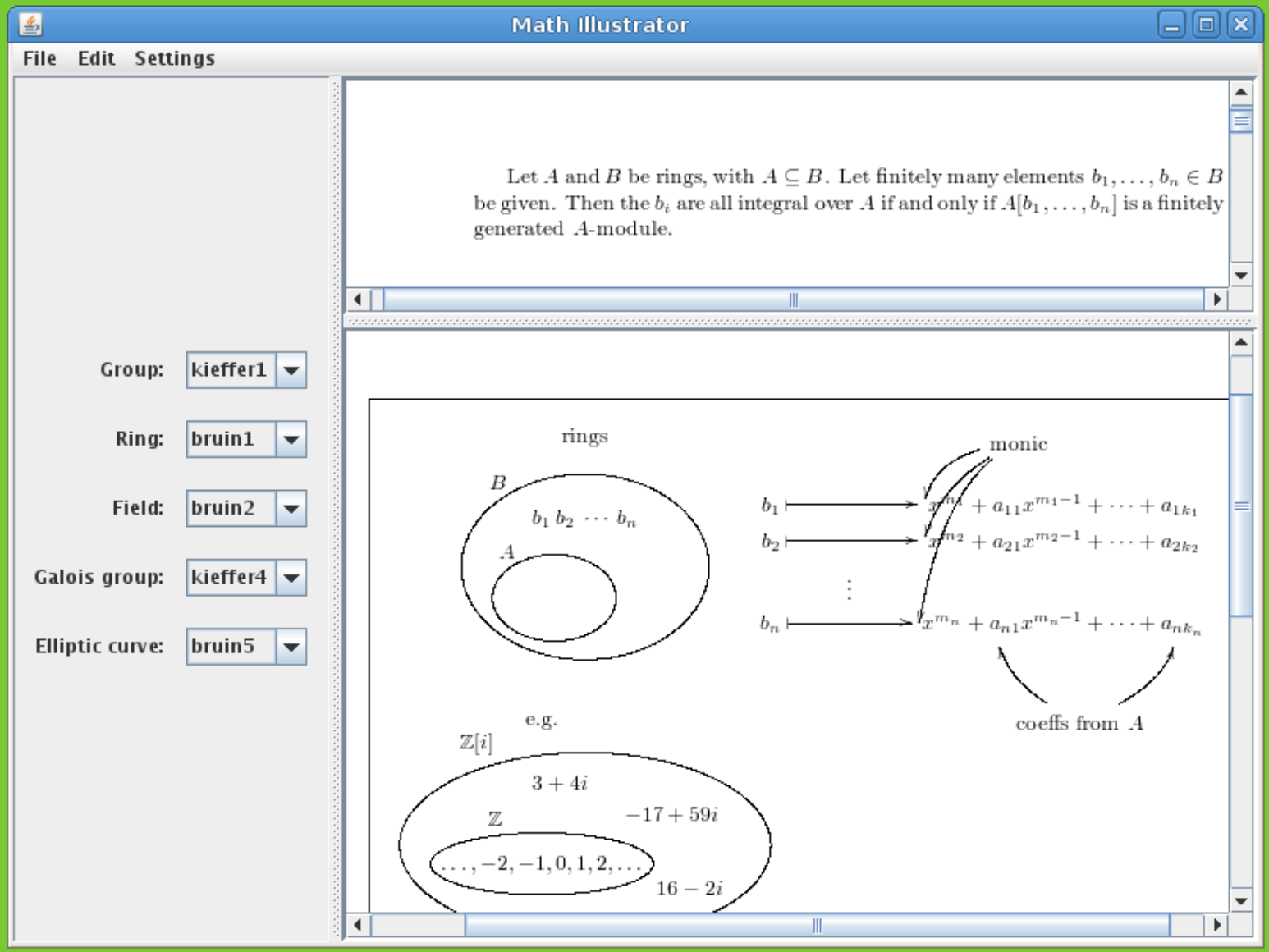
into this

Let A and B be rings, with $A \subseteq B$. Let finitely many elements $b_1, \ldots, b_n \in B$ be given. Then the b_i are all integral over A if and only if $A[b_1, \ldots, b_n]$ is a finitely generated A-module.

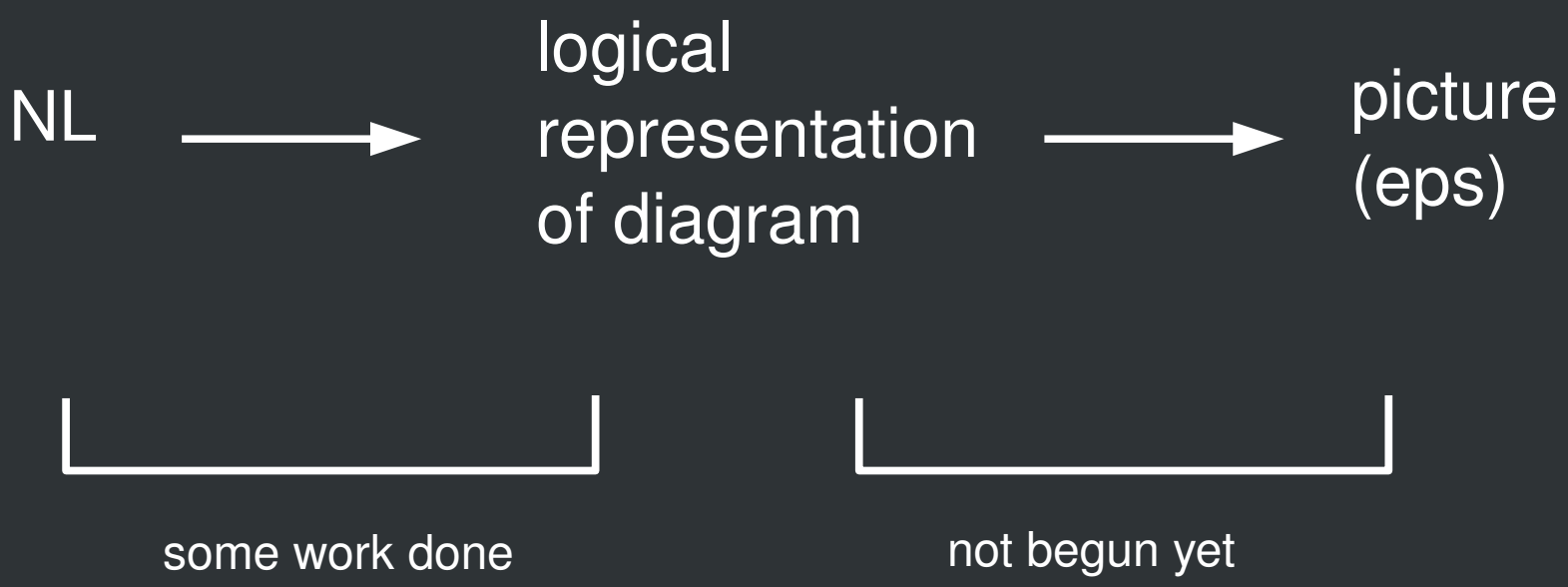
automatically.



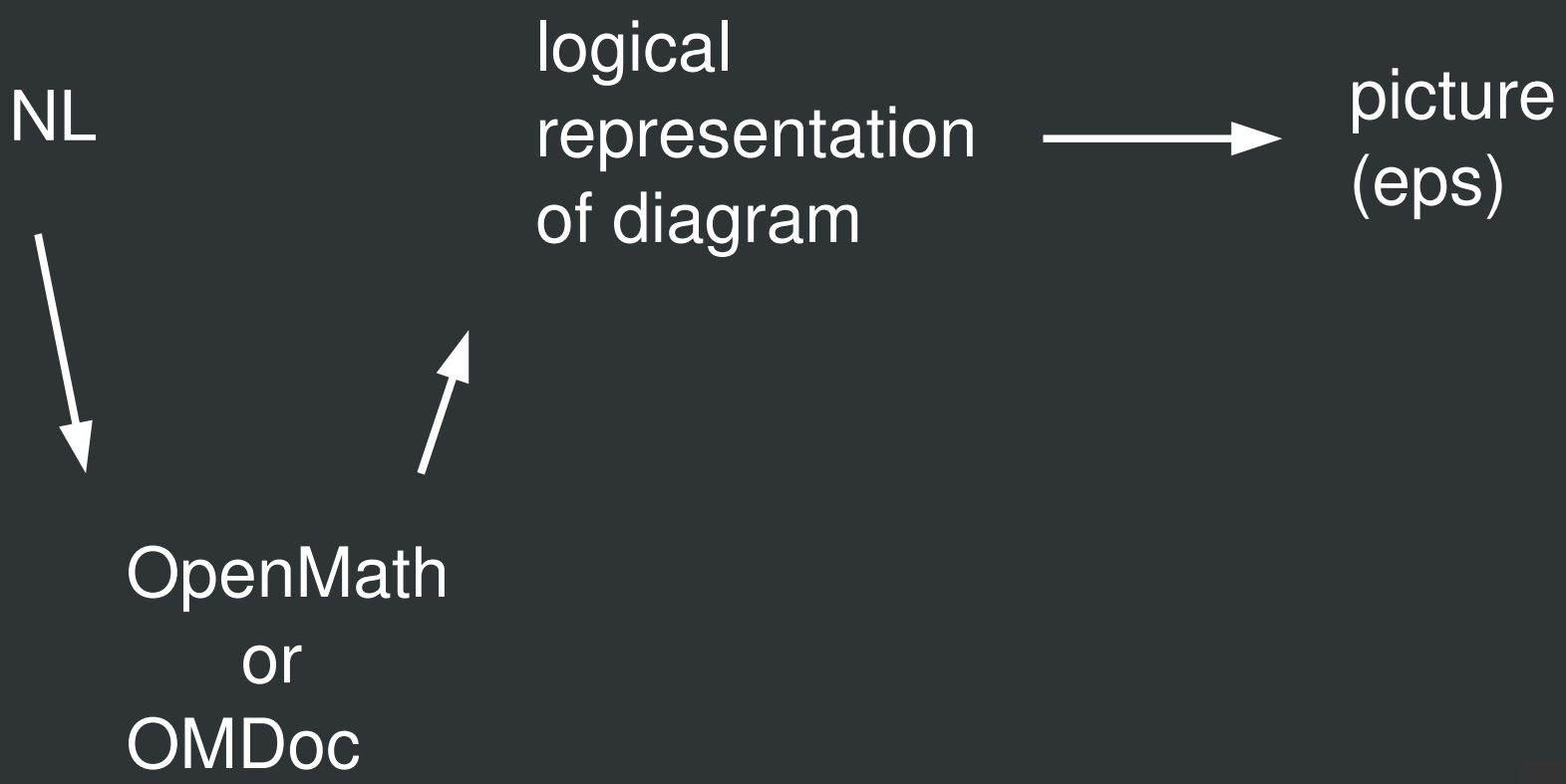
Deployment: GUI



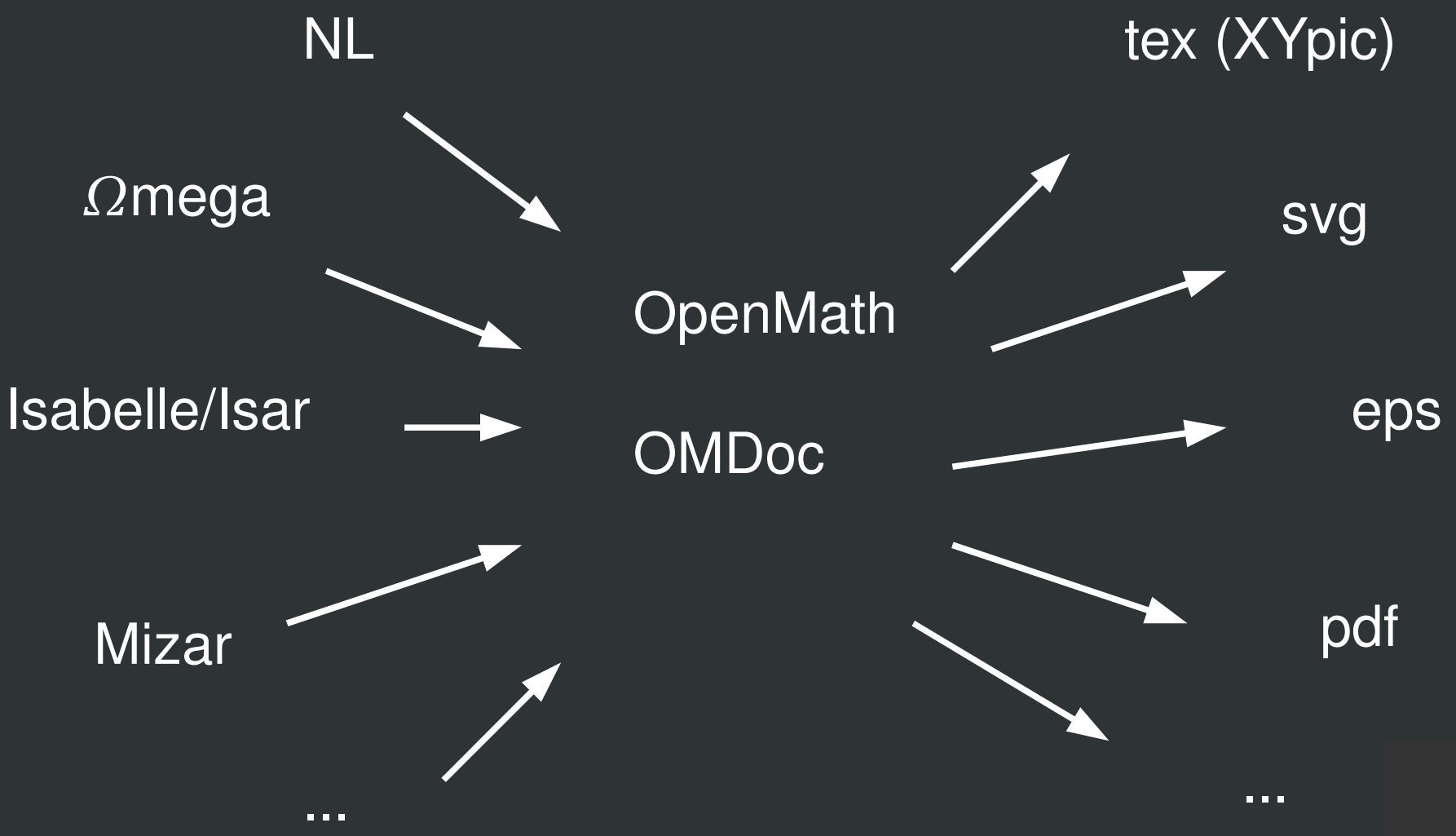
Modularity


NL
logical representation of diagram
picture (eps)
some work done
not begun yet

Modularity, future



Future generalization



NL → LDR (logical diagram representation)

- Unit of input: a "paragraph"

Theorem statement in full:

Let A and B be rings, with $A \subseteq B$. Let finitely many elements $b_1, \ldots, b_n \in B$ be given. Then the b_i are all integral over A if and only if $A[b_1, \ldots, b_n]$ is a finitely generated A-module.

One paragraph in a proof?

Since $\Omega = \cup_{n \geq 1} \mathbb{Q}(\mu_n)$, we find $G(\Omega|\mathbb{Q}) = \lim_{\leftarrow n} G(\mathbb{Q}(\mu_n)|\mathbb{Q}) = \lim_{\leftarrow n} (\mathbb{Z}/n\mathbb{Z})^* = \hat{\mathbb{Z}}^*$. But $\hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$, and $\mathbb{Z}_p^* \cong \mathbb{Z}_p \times \mathbb{Z}/(p-1)\mathbb{Z}$ for $p \neq 2$ and $\mathbb{Z}_2^* \cong \mathbb{Z}_2 \times \mathbb{Z}/2\mathbb{Z}$. Consequently, $G(\Omega|\mathbb{Q}) \cong \hat{\mathbb{Z}}^* \cong \hat{\mathbb{Z}} \times \hat{T}$, where $\hat{T} = \prod_{p \neq 2} \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. This shows that the torsion subgroup T of $G(\Omega|\mathbb{Q})$ is isomorphic to the torsion sub-

Thm lang. vs. Pf lang.

- Theorem language:
  - Let / Assume / Suppose ...
  - ... given
  - Then ...

Let A and B be rings, with $A \subseteq B$. Let finitely many elements $b_1, \ldots, b_n \in B$ be given. Then the b_i are all integral over A if and only if $A[b_1, \ldots, b_n]$ is a finitely generated A-module.

Thm lang. vs. Pf lang.

Since $\Omega = \cup_{n\geq 1}\mathbb{Q}(\mu_n)$, we find

$$G(\Omega|\mathbb{Q}) = \lim_{\leftarrow n} G(\mathbb{Q}(\mu_n)|\mathbb{Q}) = \lim_{\leftarrow n}(\mathbb{Z}/n\mathbb{Z})^* = \hat{\mathbb{Z}}^*.$$

But $\hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$, and $\mathbb{Z}_p^* \cong \mathbb{Z}_p \times \mathbb{Z}/(p-1)\mathbb{Z}$ for $p \neq 2$ and $\mathbb{Z}_2^* \cong \mathbb{Z}_2 \times \mathbb{Z}/2\mathbb{Z}$.

Consequently, $G(\Omega|\mathbb{Q}) \cong \hat{\mathbb{Z}}^* \cong \hat{\mathbb{Z}} \times \hat{T}$, where $\hat{T} = \prod_{p\neq 2} \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

This shows that

the torsion subgroup T of $G(\Omega|\mathbb{Q})$ is isomorphic to the torsion subgroup of $\hat{T}$.

Since the latter contains the group $\bigoplus_{p\neq 2} \mathbb{Z}/(p-1)\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, we see that the closure $\overline{T}$ of T is isomorphic to $\hat{T}$. Now, if $\tilde{\mathbb{Q}}$ is the fixed field of T, this implies that $G(\tilde{\mathbb{Q}}|\mathbb{Q}) = G(\Omega|\mathbb{Q})/\overline{T} \cong \hat{\mathbb{Z}}$.

That is, ...

Since A, we find B. But C, and D and E.
Consequently, F, where G. This shows that H.
Since I, we see that J. Now, if K, this implies that L.

Thm lang. vs. Pf lang.

Since A, we find B. But C, and D and E.
Consequently, F, where G.
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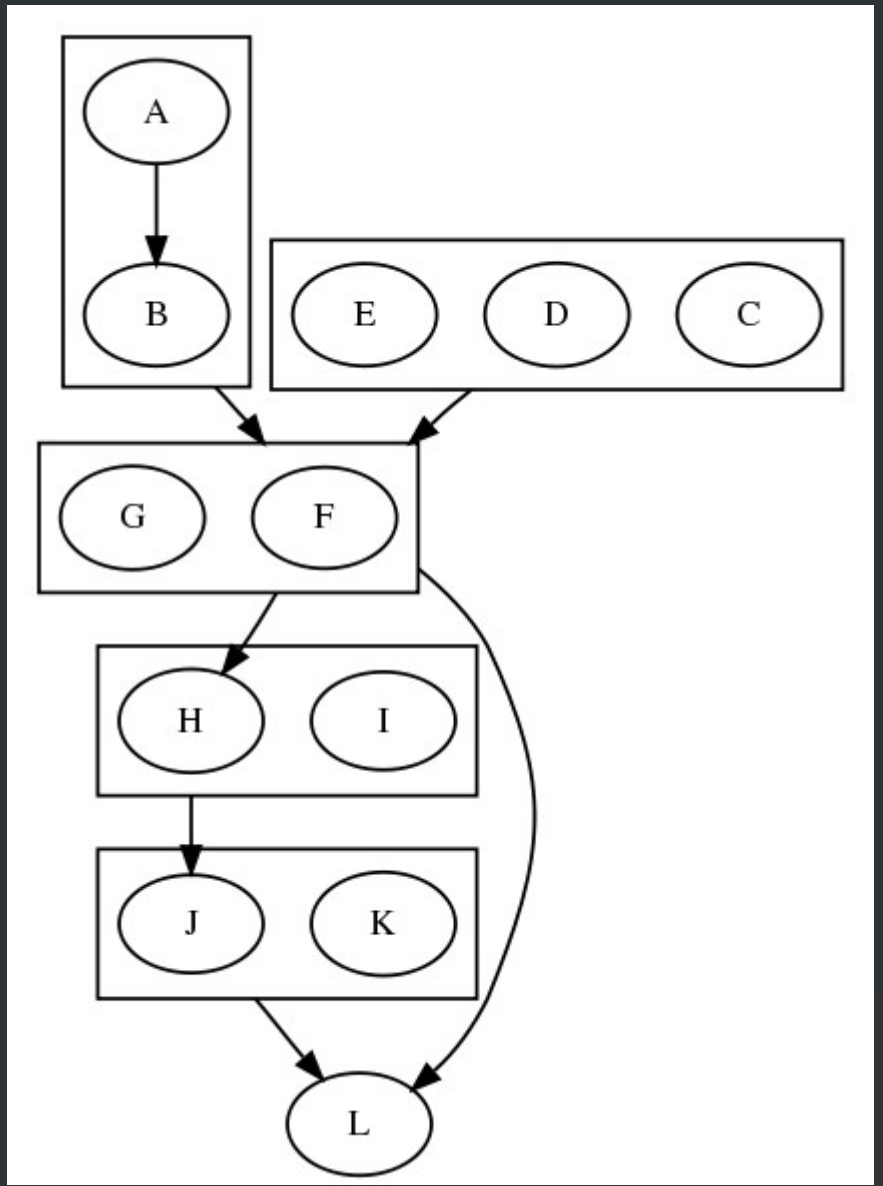
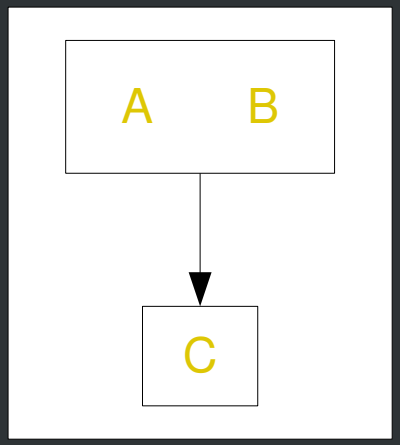
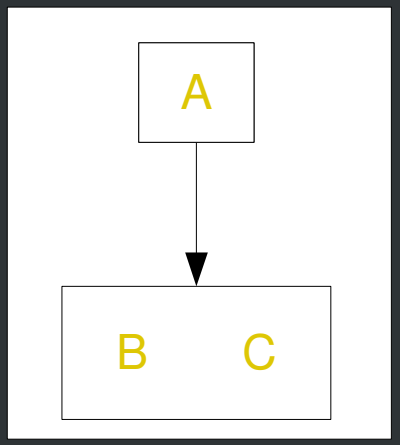


Propositional chunking

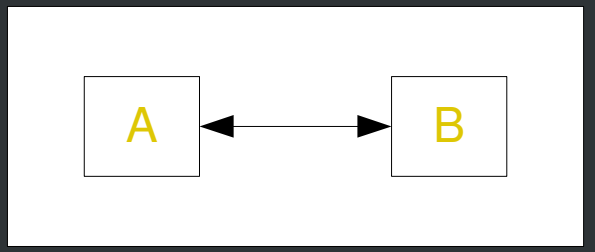
$A \wedge B \rightarrow C$



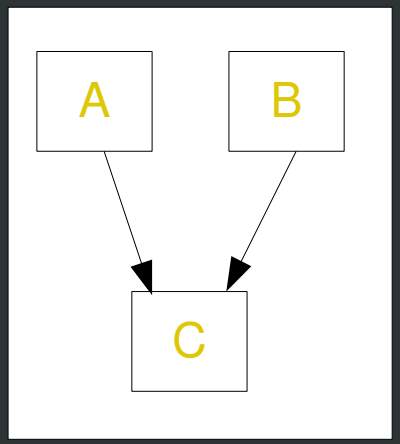
$A \rightarrow B \wedge C$



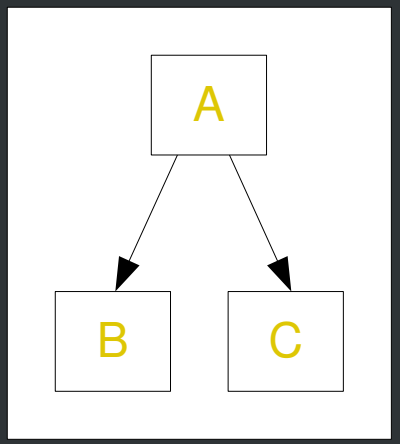
$A \leftrightarrow B$



$A \vee B \rightarrow C$

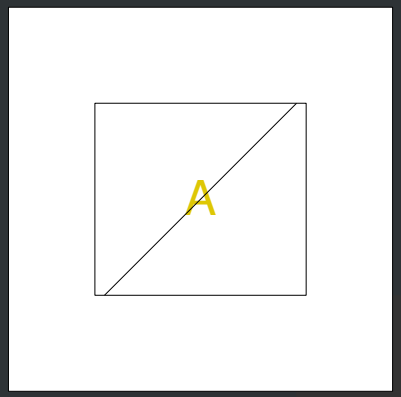


$A \rightarrow B \vee C$?



$A \wedge \neg B \rightarrow C$?

$\neg A$?

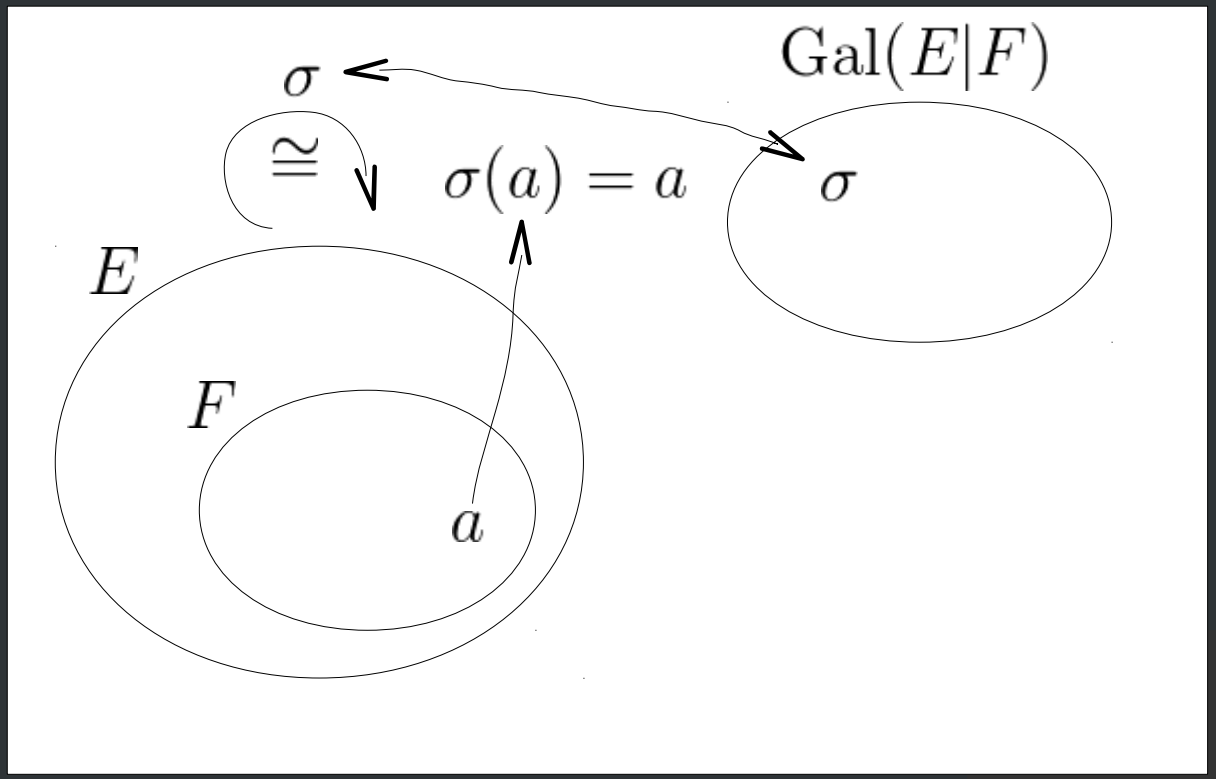


NL → LDR

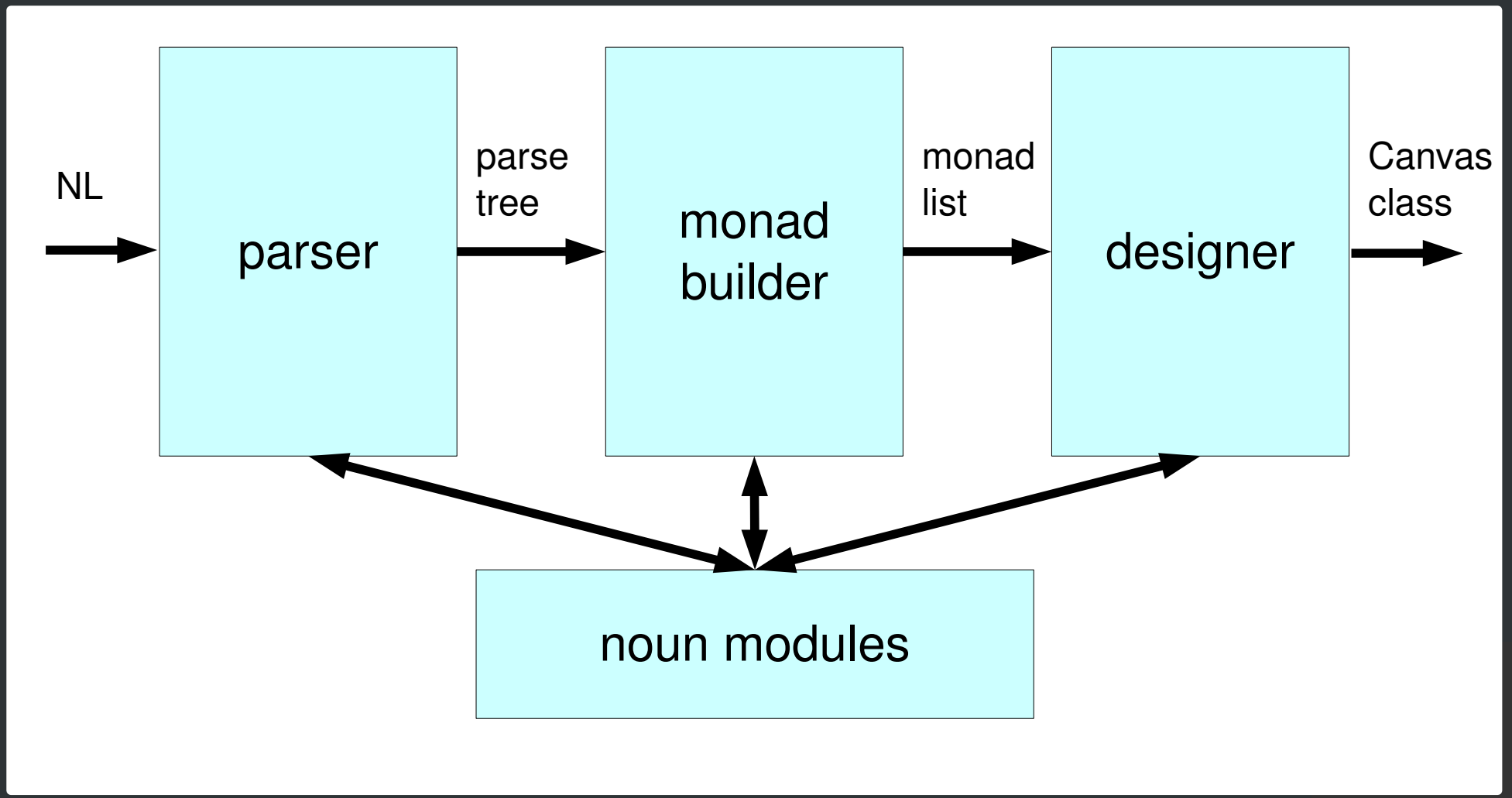
Input:

Let $E|F$ be fields, and let G be the Galois group of E over F.

Output:
a logical representation of a drawing like this

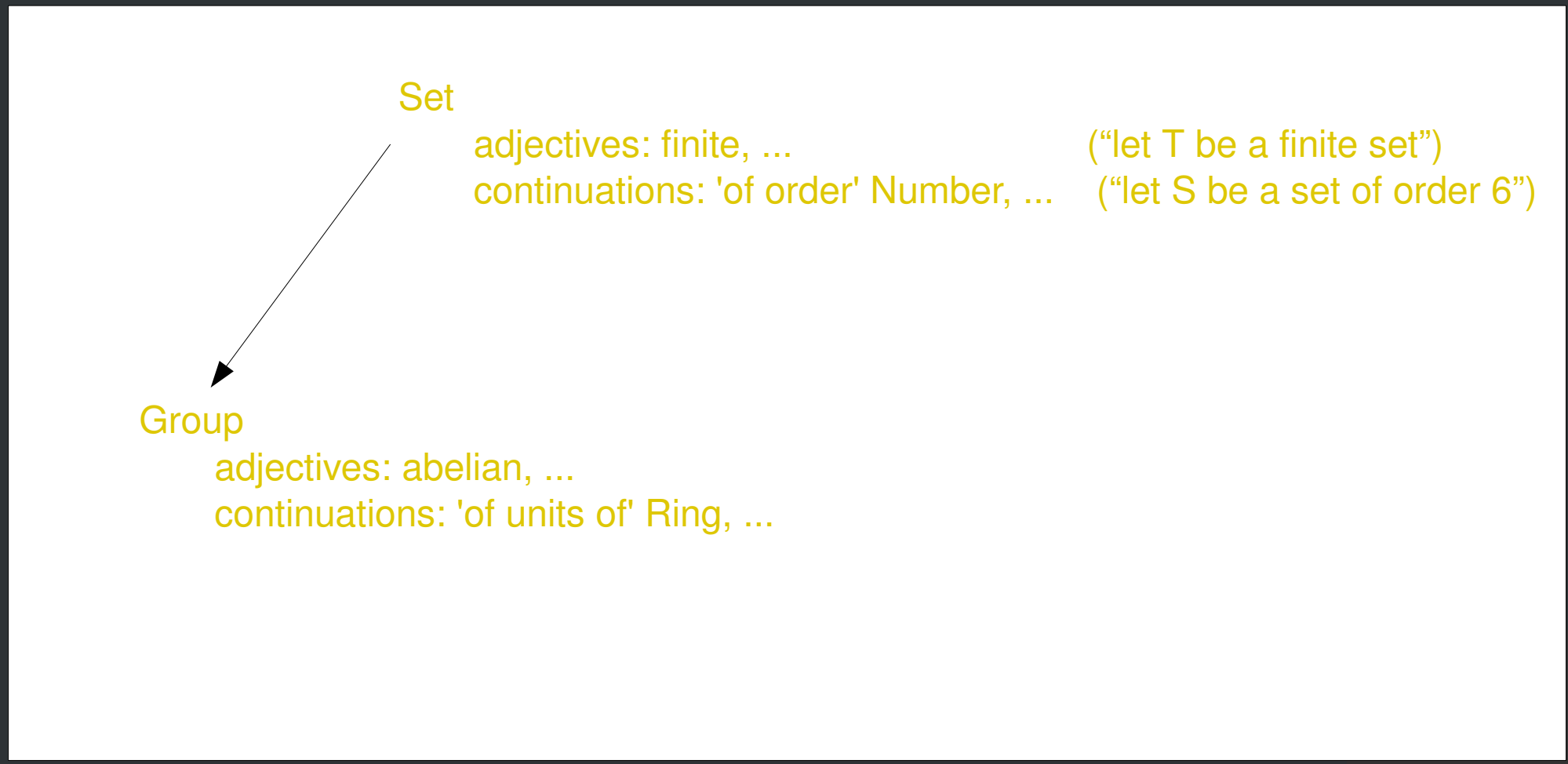


Architecture

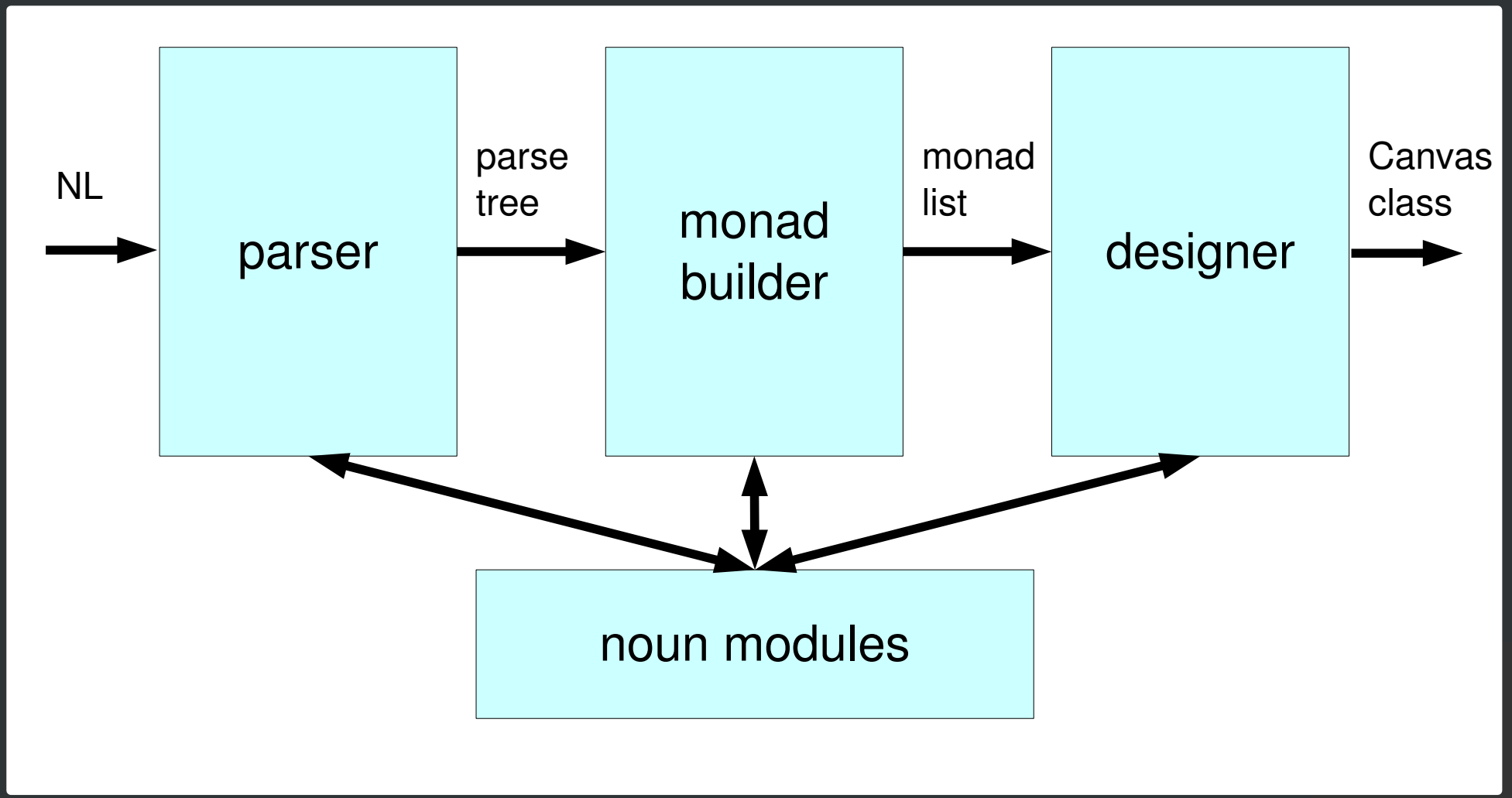


noun modules

- nouns are classes in a hierarchy



Architecture



parser

```
testSens = {
    2:'Let a group $G$ be given.',
    3:'Let $G$ be a group.',
    4:'Let $G$ and $H$ be groups.',
    5:'Let a group $G$ and a set $S$ be given.',
    6:'Let $G$ be the Galois group of $E$ over $F$.',
    7:'If $G$ is a cyclic group, then $G$ is a group.',
    8:'If $G$ is a cyclic group, then $G$ is abelian.',
    9:'Suppose $G_1$ and $G_2$ are groups.',
   10:'Suppose that $G_1,\\ldots,G_n$ are all groups.',
   11:'Assume that $G$ is abelian.',
   12:'Let $E|F$ be an extension field.',
   13:'Suppose $G$ is a group, $H$ is a subgroup, and $n$ is its index.',
   14:'Let $U$ in $\\mathscr{T}$ be given.',
   15:'Let $g$ and $h$ in a group $G$ be given, where $G$ is Abelian.',
   16:'Let $g \\in G$ be given, where $G$ is a group.',
   17:'Let fields $E|F$ be given, and let $G$ be the Galois group of $E$, over $F$.',
   18:'Let $E|F$ be fields, and let $G$ be the Galois group of $E$ over $F$.'
}
```

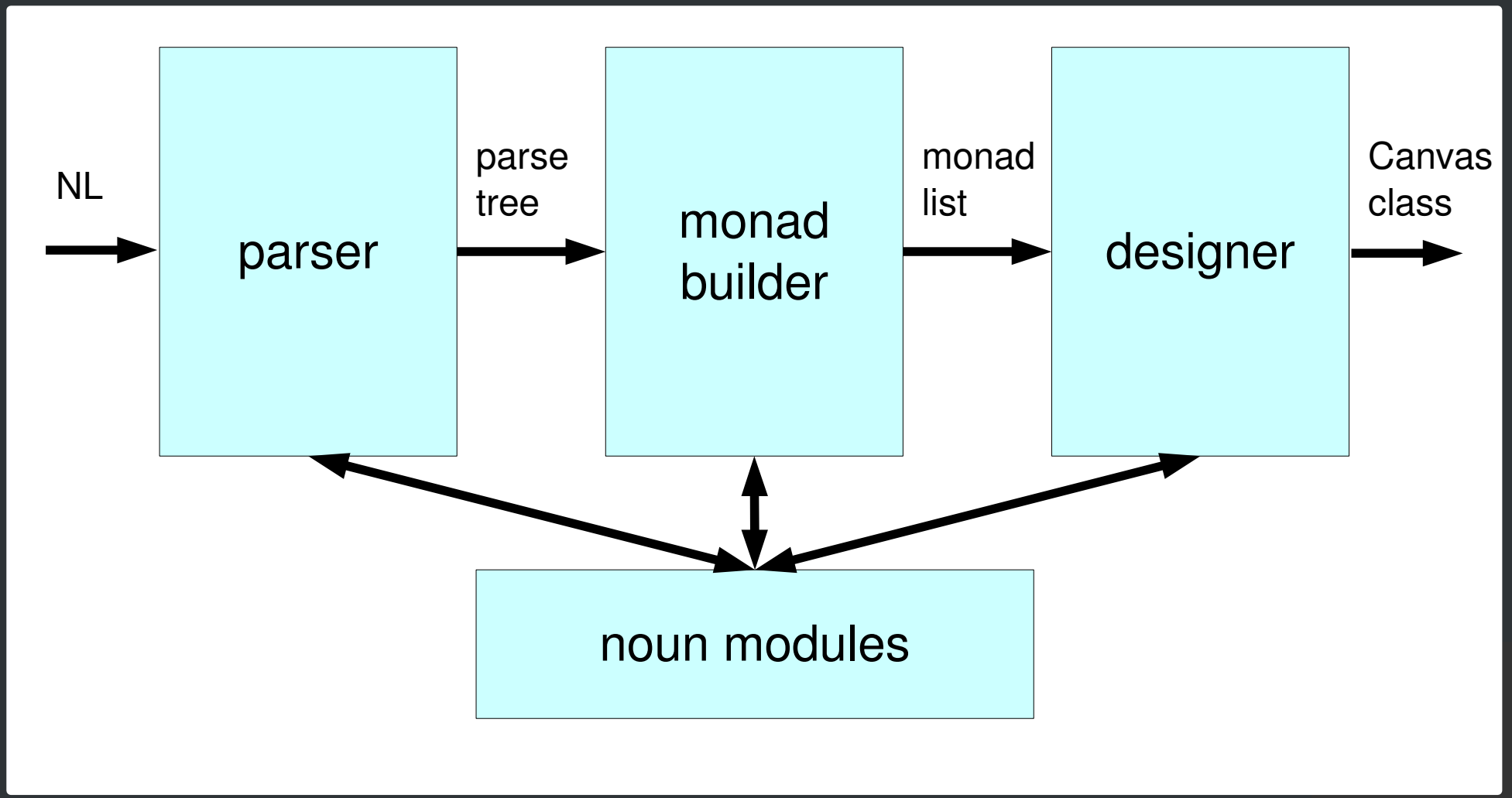
parser

```
SENcore ::=   Subject CPL (NP|PN|PA|'given')
            | SymbNPIRNP
            | NP 'consists of' NP

ASSER ::= (   'for' NP (SubjConj NP)* (COM)? ASSER
            | 'there' ('exists'|'exist'|'is'|'are') NP (SubjConj NP)*
            | SENcore (Conj SENcore)*
          )
          (COM)?
          (   ('where'|'with'|'for'|'such that') ASSER
            | 'and' ASSER
            | 'or' ASSER
          )?
```

parser

```
SEN ::= (  'let' SENcore
         | ('assume'|'suppose') ('that')? ASSER
         | 'if' ASSER (COM)? 'then' ASSER
         | 'then' SEN
         | ASSER
        )
        (COM)?
        (  ('where'|'with'|'for'|'such that') ASSER
         | 'and' SEN
         | 'or' SEN
         | 'if and only if' SEN
         | 'implies' ('that')? SEN
        )?
```

Architecture



monad builder

Let $E|F$ be fields, and let G be the Galois group of E over F.

```
MonadList:

Monad:
    Symbol: G
    Types: Set([('GaloisGroup.GaloisGroup', ':7:0')])
    Relations: Set([])

Monad:
    Symbol: E
    Types: Set([('Field.ExtensionField', ':1:0')])
    Relations: Set([('\\subset', 2, ('F',), ':1:0')])

Monad:
    Symbol: F
    Types: Set([('Field.Field', ':1:0'), ('Set.Set', ':1:0')])
    Relations: Set([('\\subset', 1, ('E',), ':1:0')])
```

“Monads”

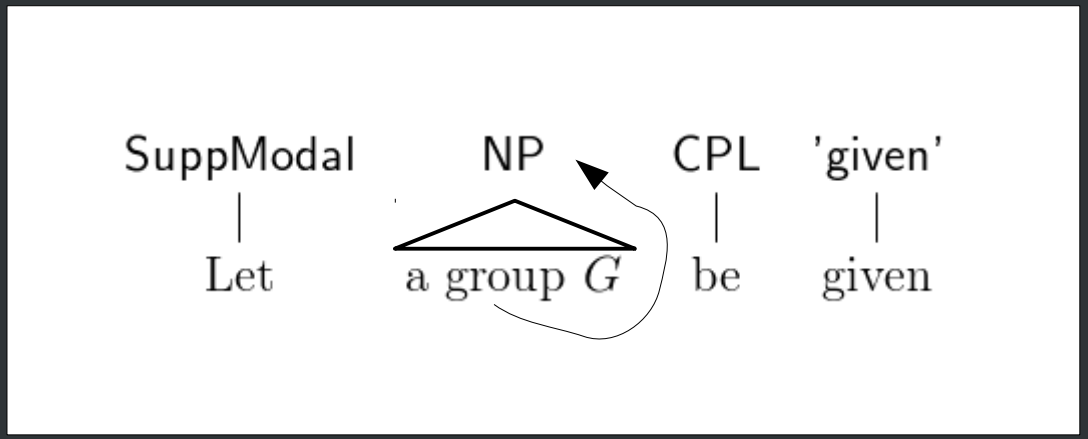
Gottfried Wilhelm von Leibniz

- 1714, *Monadologie*
- Every relation into which an object enters is regarded as a property of that object.

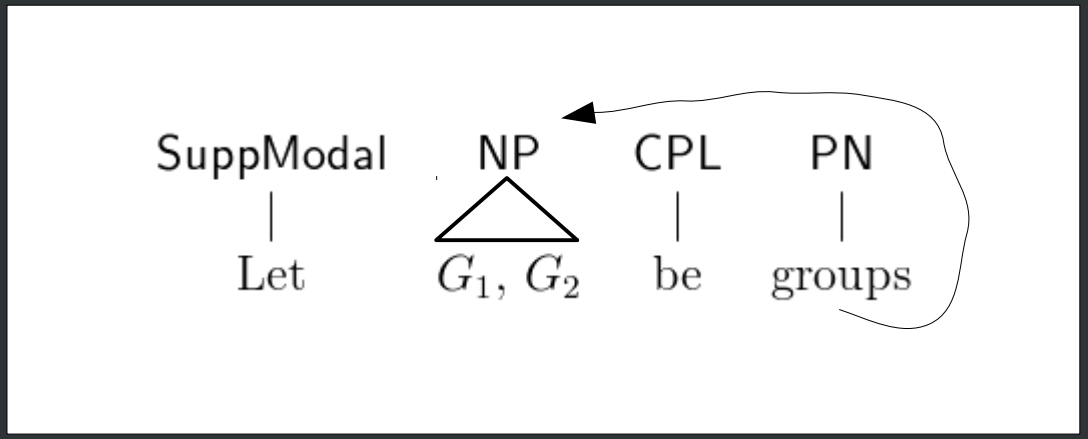
monad builder algorithm

```
def mainloop(tree):
    timeStampNPs(tree)
    ML = MonadList()
    changed = True
    while changed:
        changed = False
        changed |= applyPNtypes(tree)
        changed |= applyMathObjectContinTypes(tree)
        changed |= extractMonads(tree,ML)
        changed |= typeSpecificMonadExtraction(tree,ML)
        changed |= monadRelnTypes(ML)
```

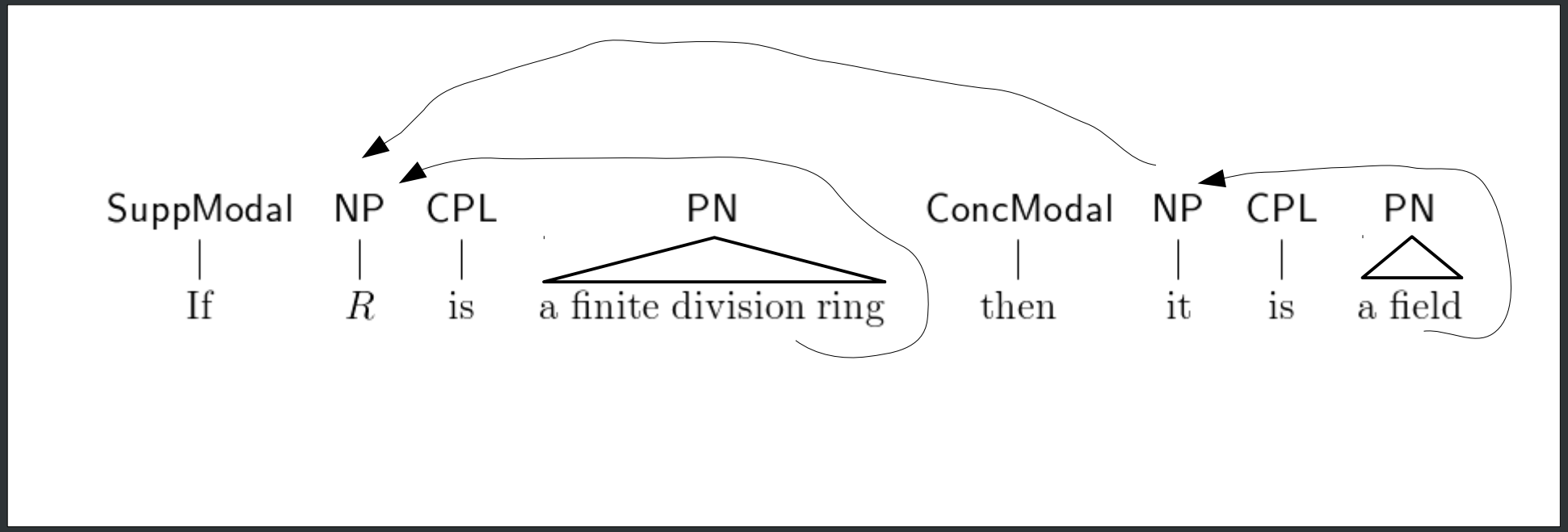
noun in NP → type for NP



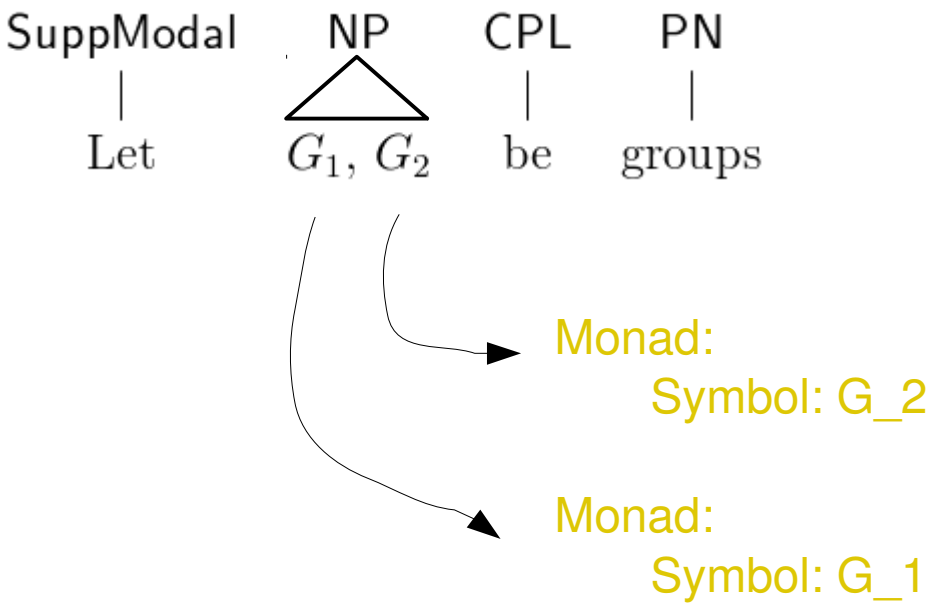
noun in PN → type for NP



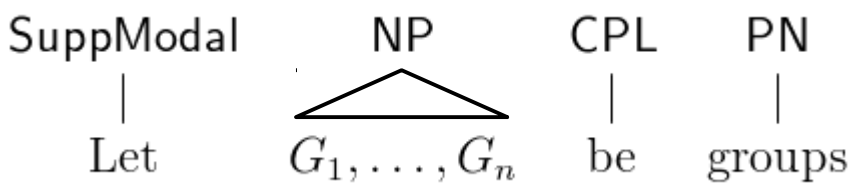
address of type info is important



extract symbols

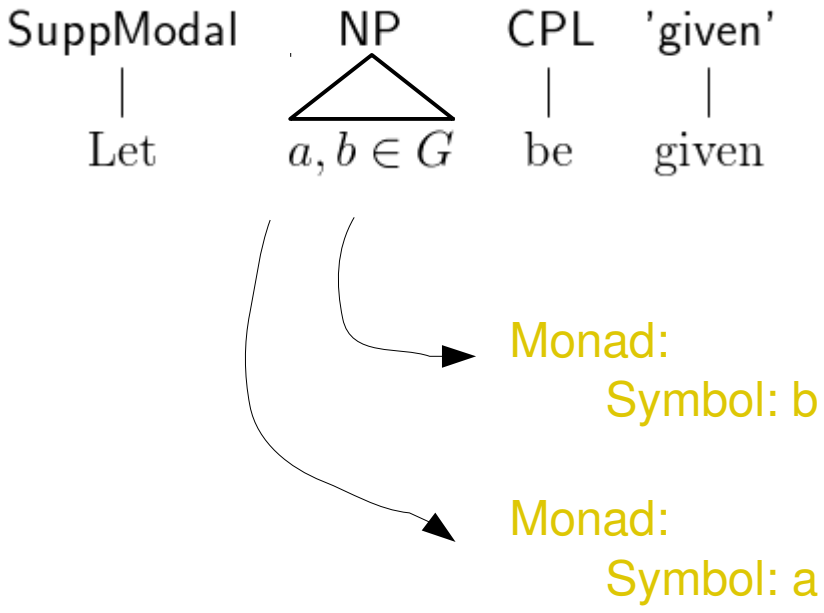


extract indeterminate list

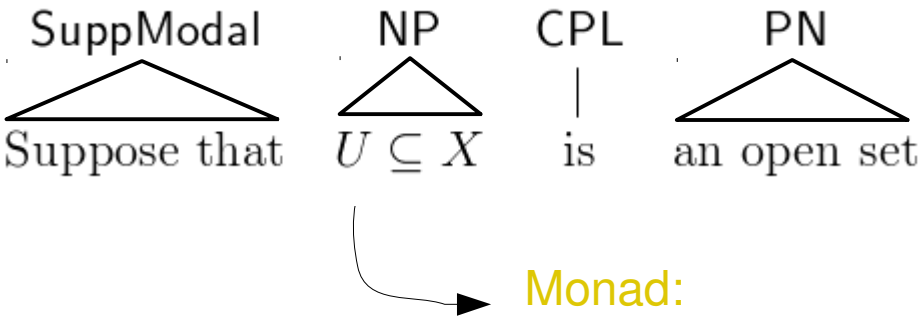


Monad:
Symbol: G_i

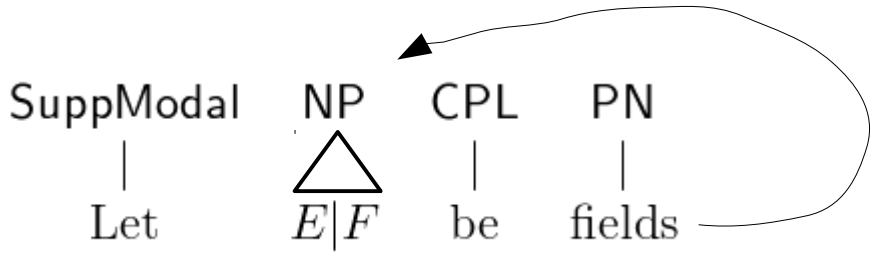
extract symbols, with “known” IR



extract symbols, with “known” IR



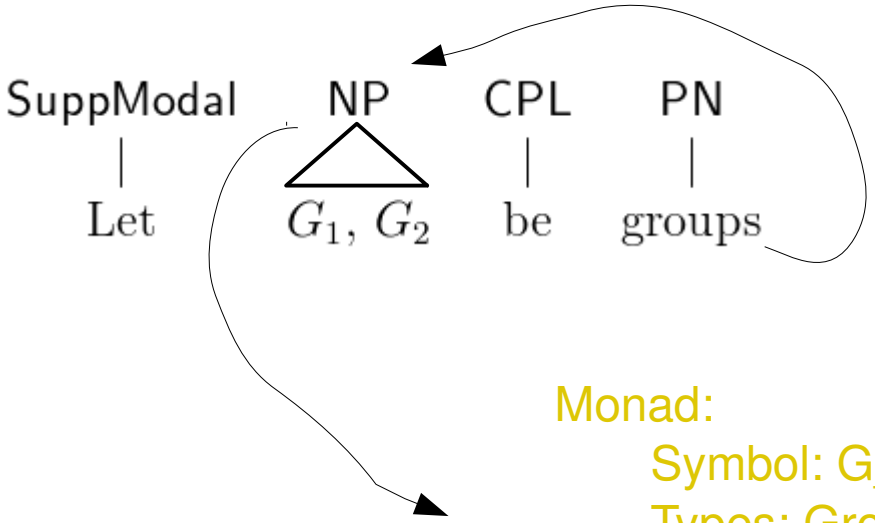
extract symbols, with “unknown” IR



Monad:
Symbol: F

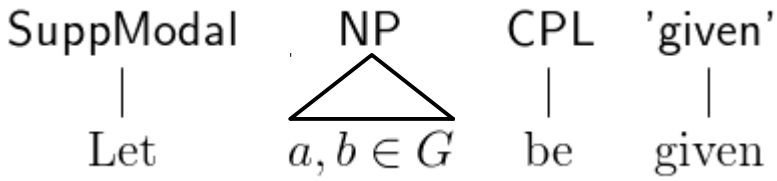
Monad:
Symbol: E

type from NP → its monads



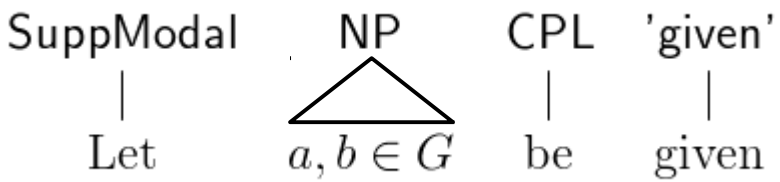
Monad:
Symbol: G_2
Types: Group
Monad:
Symbol: G_1
Types: Group

compute functional type, for “known” IR



Monad:
Symbol: b
Type: GroupElement
Monad:
Symbol: a
Type: GroupElement

add relation to monads

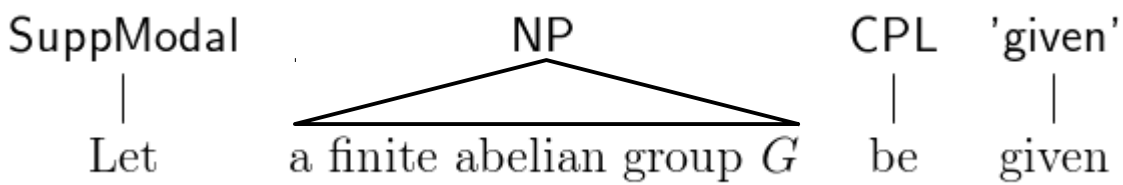


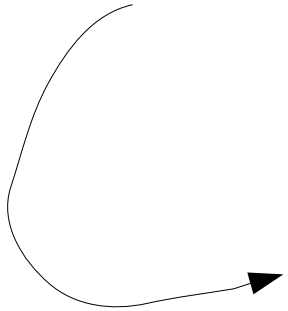
Monad:
- Symbol: G
- Relations: (set_memb, 2, a), (set_memb, 2, b)

Monad:
- Symbol: a
- Relations: (set_memb, 1, G)

Monad:
- Symbol: b
- Relations: (set_memb, 1, G)

pass adjectives from NP to monads





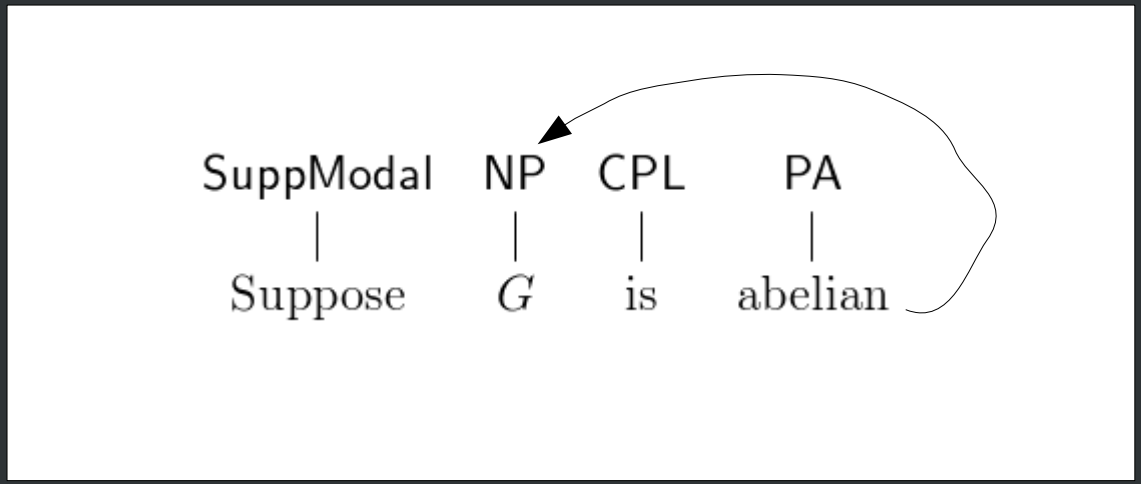
Monad:

Symbol: G

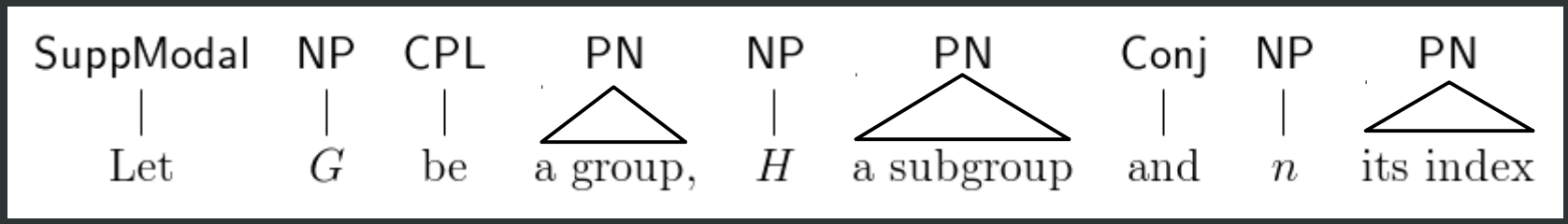
Types: Group

Adjectives: [(abelian, Group),
(finite, Set)]

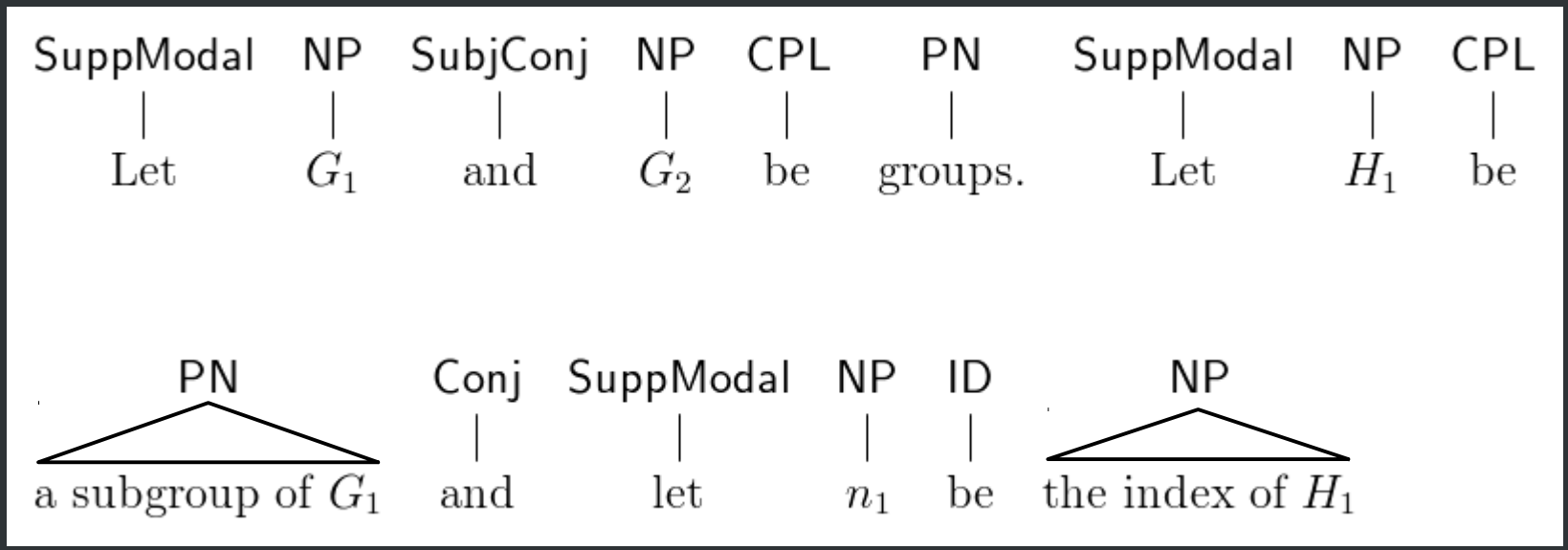
pass adjective from PA to NP



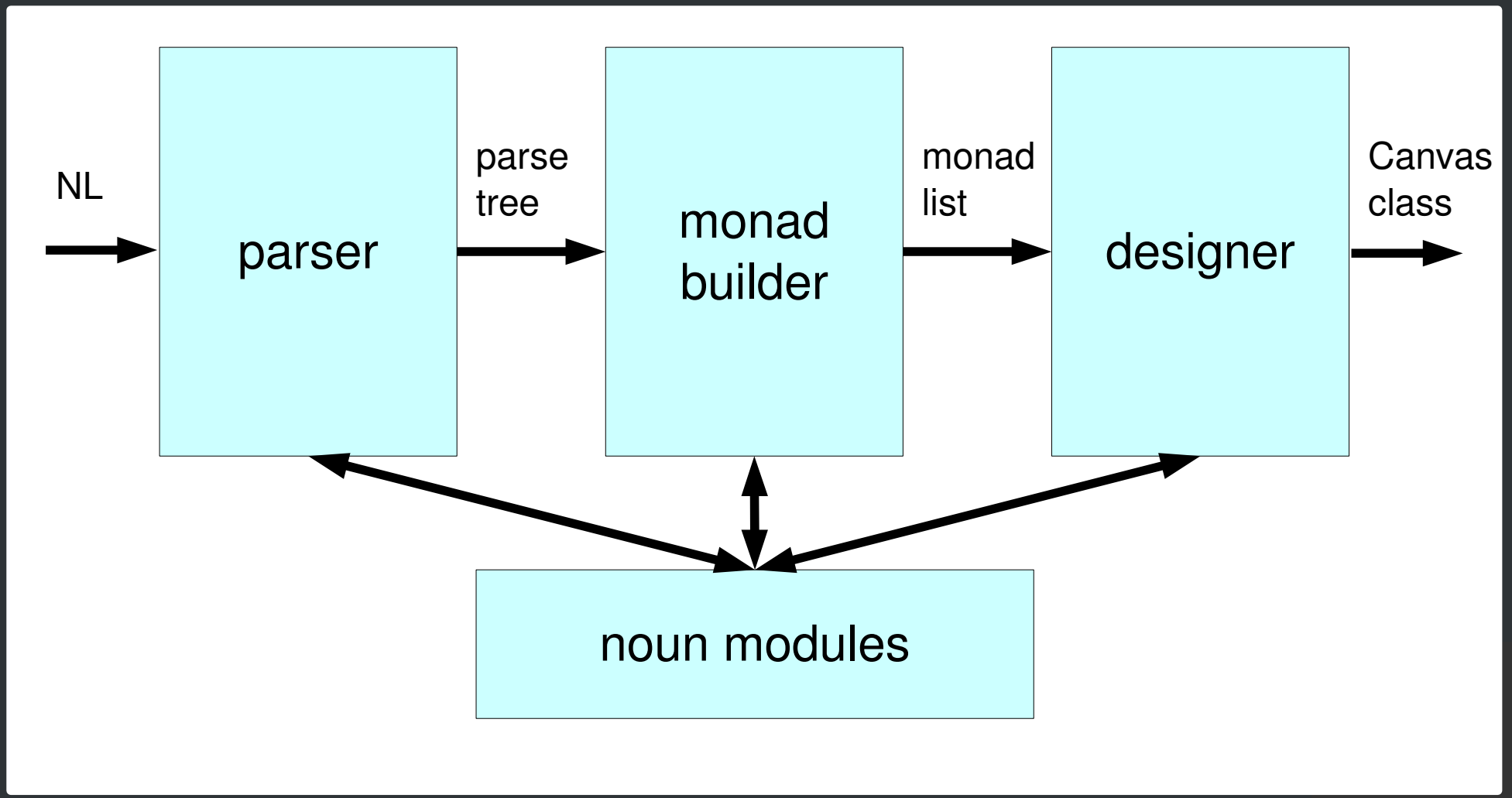
Context sensitivity



Context sensitivity



Architecture

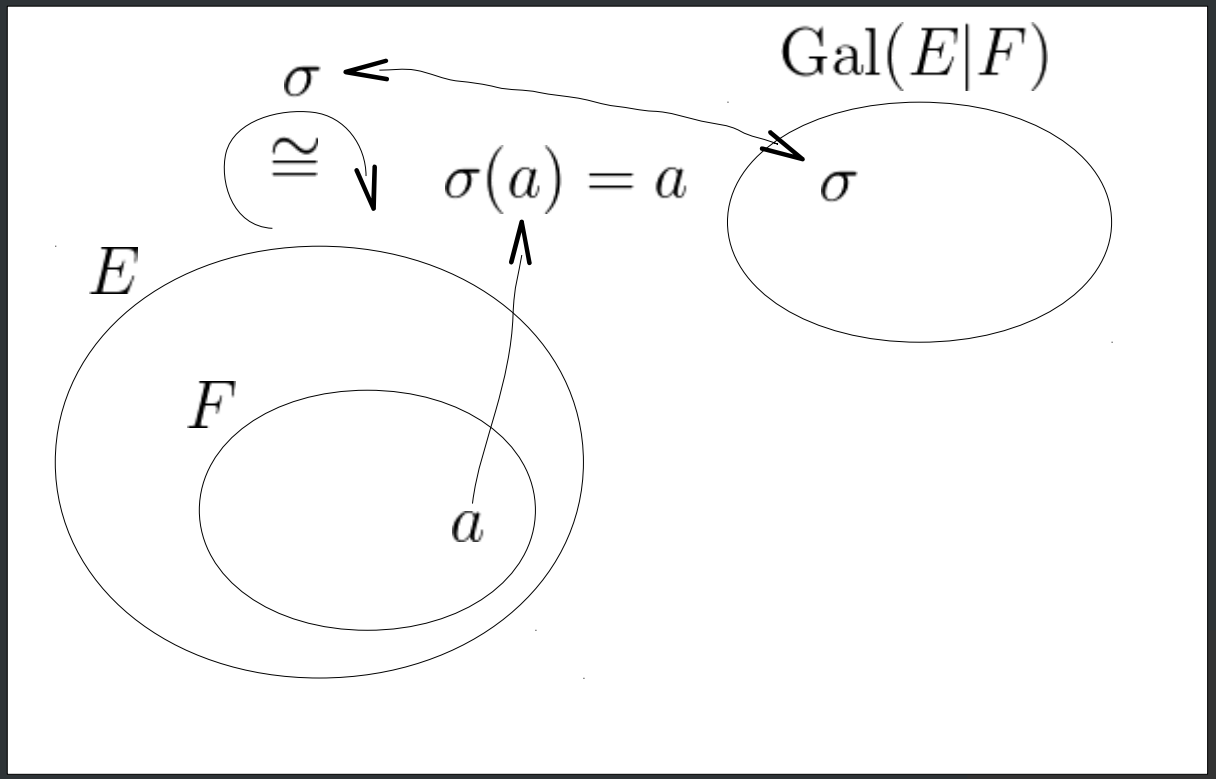


NL → LDR

Input:

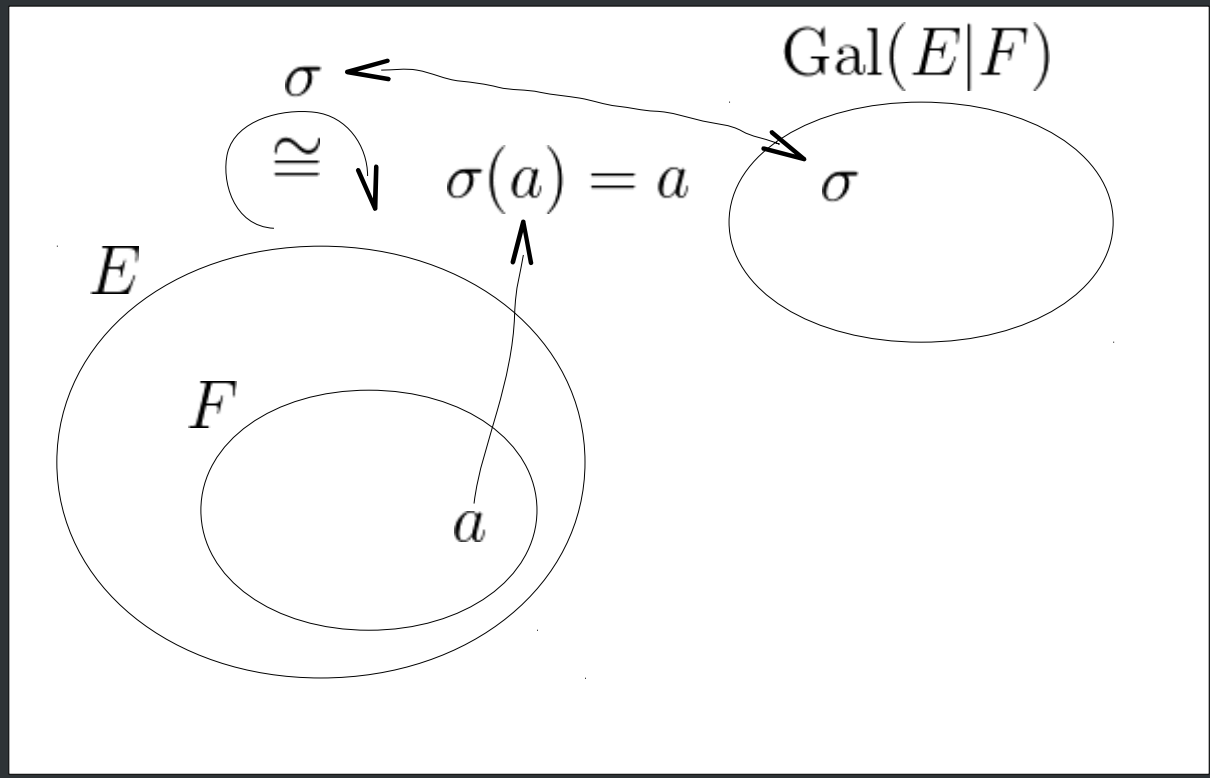
Let $E|F$ be fields, and let G be the Galois group of E over F.

Output:
a logical representation of a drawing like this

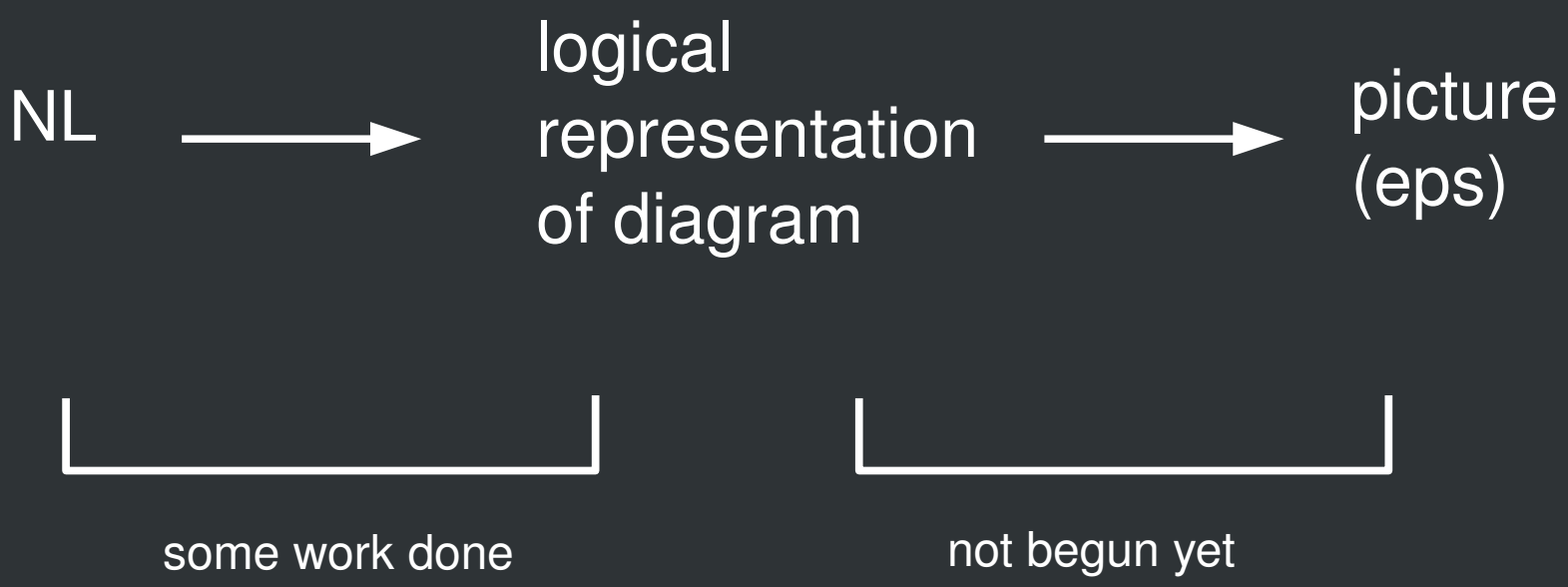


NL → log. rep.

```
def generalPicture():
    E.generalPicture()
    F.generalPicture()

    G = carrier(self)
    G.label('Gal(%s|%s)' % (E,F))
    sigma = G.add_element(
        self.preferred_names
    )

    automorph = E.add(ENDOMAP)
    upperlabel = automorph.label(
        sigma,ABOVE
    )
    automorph.label(r'\cong',BELOW)

    bidirectional(upperlabel,sigma)

    F.add_callout(
        'a',
        '%s(a) = a' % sigma,
        sigma.NEAR
    )
```



Modularity


NL
logical representation of diagram
picture (eps)
some work done
not begun yet

log. rep. → image

- Pyscript: http://pyscript.sourceforge.net