The Vandermonde Matrix and the Discriminant, by Brute Force

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For integer $n \geq 2$, let V_n be the Vandermonde matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ u_1 & u_2 & \cdots & u_n \\ u_1^2 & u_2^2 & \cdots & u_n^2 \\ \vdots & \vdots & & \vdots \\ u_1^n & u_2^n & \cdots & u_n^n \end{bmatrix}$$

and let Δ_n be the product

$$\Delta_n = \prod_{i<j} (u_j - u_i),$$

whose square is the discriminant of the u_i.

There is a very nice and easy proof by induction that

$$\Delta_n = |V_n|, \tag{1}$$

which uses elementary row and column operations in the inductive step, to reduce both the first row and first column of the Vandermonde matrix to all zeros except a single 1 where this row and column meet. In a few more steps, the proof is done.

I offer instead a "brute force" proof which insists that it should be possible to see combinatorially that all the necessary cancellation takes place in the expansion of the product Δ_n. The proof is by induction, the base step $n = 2$ being obvious.

Instead of doing the induction step in generality, we demonstrate it for the case in which we pass from $n = 4$ to $n = 5$.

We have

$$\Delta_5 = (u_5 - u_1)(u_5 - u_2)(u_5 - u_3)(u_5 - u_4)\,\Delta_4 \tag{2}$$

$$= (u_5 - u_1)(u_5 - u_2)(u_5 - u_3)(u_5 - u_4) \sum_{\pi} (-1)^{|\pi|}\, u_4^{\pi(3)} u_3^{\pi(2)} u_2^{\pi(1)} u_1^{\pi(0)} \tag{3}$$

where π ranges over all permutations $\pi : \{3, 2, 1, 0\} \to \{3, 2, 1, 0\}$, and $|\pi|$ denotes the parity of π. Here we have used the induction hypothesis, and the definition of the determinant of a matrix (a_{ij}) as $\sum_{\varphi} (-1)^{|\varphi|} \prod_i a_{i\varphi(i)}$. To complete the proof, again using this definition of the determinant, we need to show that in the formal expansion of the product (3), those terms that survive cancellation are precisely those in which the exponents of $u_1, u_2, \ldots, u_5$ are all distinct.

In the formal expansion of (3), each term is obtained by first choosing one term from each of the first four factors, and then choosing a permutation π. The choice over the first four factors can be represented by a binary vector

$$r = (r_3, r_2, r_1, r_0) \in \{0,1\}^4$$

in which r_i is 0 if and only if you chose the term u_5 in the factor $(u_5 - u_{i+1})$. Thus, the ordered pair (r, π) shall represent the term:

$$(-1)^{|\pi|+|r|}\, u_5^{4-|r|} u_4^{\pi(3)+r_{\pi(3)}} u_3^{\pi(2)+r_{\pi(2)}} u_2^{\pi(1)+r_{\pi(1)}} u_1^{\pi(0)+r_{\pi(0)}}. \tag{4}$$

If on the other hand the general term τ is written $\tau = (-1)^s\, u_5^{a_5} u_4^{a_4} \cdots u_1^{a_1}$, then our goal is to show that those terms in which the a_i are all distinct survive cancellation, whereas all other terms cancel out. This will complete the proof.

Our first observation is that the general term (4) is completely determined by just the exponents a_4, a_3, a_2, a_1. Furthermore, all five of the a_i are distinct if and only if these four are: In the forward direction, this is obvious; in the backward direction, it follows from the fact that $\Sigma a_i = \binom{5}{2}$ and that $0 \le a_i < 5$.

Now, every list of exponents (a_4, a_3, a_2, a_1) occurring in the formal expansion of the RHS of (3) is obtained by first adding each of the 2^4 binary vectors $r \in \{0,1\}^4$ to the vector $(3, 2, 1, 0)$, which yields

$$\begin{matrix} 3210 & 3211 & 3220 & 3221 \\ 3310 & 3311 & 3320 & 3321 \\ 4210 & 4211 & 4220 & 4221 \\ 4310 & 4311 & 4320 & 4321 \end{matrix} \tag{5}$$

and then writing down each of the 4! formal permutations of each of these exponent vectors. Note that if you pick any two vectors v, w from this table, they are composed of a different collection of numbers, so that no permutation of v is equal to any permutation of w. (It is easy to prove that this generalizes from the case of $n = 4$ to the general case. Namely, prove it by induction on n, using the observation that any vector in the table can contain n at most once, in the induction step.) Thus, cancellation can only take place within the set of permutations of a single one of these vectors.

If you pick a vector from the table (5) in which all four elements are distinct, then clearly all 4! permutations of this vector are distinct from one another. Thus, those terms in which the a_i are all distinct survive cancellation, as claimed.

Finally, pick any vector v from table (5) in which some number appears more than once. Our first observation is that in this case this number must appear exactly twice. (This is an obvious consequence of the way in which the entries of the table were generated, namely, by adding a binary vector to a sequence of consecutive integers.) Thus, reordering if necessary, v can be written as

$$b_1 b_1 b_2 b_2 \cdots b_k b_k c_1 c_2 \cdots c_\ell \tag{6}$$

(here we pass from the case $n = 4$ to the general case $n = 2k + \ell$). Then we may fix a certain permutation π such that the vector v is obtained from (6) under each of the permutations

$$\pi \tau_1^{e_1} \tau_2^{e_2} \cdots \tau_k^{e_k},$$

where τ_i is the transposition swapping positions $2i$ and $2i - 1$, and where this exponent vector $(e_1, e_2, \ldots, e_k)$ runs over all possible length-k binary vectors in $\{0,1\}^k$. But it is clear that precisely half of these permutations are odd, and the other half are even, so that all these permutations of v correspond to terms (4) that cancel each other out, in the expansion of (3), as was to be shown. $\square$